# Pattern dynamics in bidimensional oscillatory media with bistable inhomogeneities

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By means of numerical simulations, we study pattern dynamics in selected examples of inhomogeneous active media described by a reaction diffusion model of the activator-inhibitor type. We consider inhomogeneities corresponding to a variation in space of the (nonlinear) reaction characteristics of the system or the diffusion constants. Three different bidimensional systems are analyzed: an oscillatory medium in a square reactor with a circular central bistable domain, and cases of a bistable stripe immersed in an oscillatory medium in a trapezoidal reactor and in a rectangular reactor with inhomogeneous diffusion. The different types of complex behavior that arise in these systems are analyzed.

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## I. INTRODUCTION

The phenomena of pattern formation are ubiquitous in almost all branches of scientific endeavor, ranging from physics and chemistry to biology and technology [1]. Among the different possible model descriptions of pattern forming systems, the reaction-diffusion models of the activator-inhibitor type [2,3] have provided a useful theoretical framework in all areas.

In a recent paper [4] we analyzed, in a unidimensional simple model, the effect of the presence of inhomogeneities in an otherwise homogeneous active system. We found a richer spread of behaviors than those occurring in typical bistable, oscillatory, or excitable homogeneous active media. The pattern formation phenomena arising in each kind of homogeneous active medium have their own characteristics [3]: in bistable media we have front structures separating quasihomogeneous domains where the system is in one of the two stable states of the uncoupled system. Examples of such structures include moving fronts in one-dimensional systems [3,5] and labyrinthine structures in bidimensional systems [6]. The most common examples of patterns arising in models of excitable media [3] are traveling pulses, spiral patterns and more complex structures of vortices in three dimensions. Those models are useful to describe chemical systems such as the Belousov-Zhabotinskii and related reactions, and also to describe biological systems such as neural or cardiac tissues [2,3,7]. In oscillatory systems, the most typical structures appearing are homogeneous oscillations, Turing patterns, and spiral patterns [1,3].

In Ref. [4], the case of a finite unidimensional oscillatory medium with an immersed bistable spot, with nonflux boundary conditions in $\pm L$, was analyzed. The reaction diffusion model considered was [8,9]

$$\dot{u} = \nabla^2 u + f(u) - v,$$

$$\dot{v} = D_v \nabla^2 v + u - \gamma v, \tag{1}$$

where $f(u) = -u^3 + u$ characterizes the nonlinearity of the medium. The inhomogeneity was introduced in the model through a spatial dependence on the parameter $\gamma$, setting $\gamma = \gamma(x) \equiv 0.9 + 5\exp(-10x^4)$. This leads to $\gamma \simeq 0.9 < 1$ for $|x| > 0.8$ (oscillatory medium) and $\gamma > 1$ for $|x| < 0.8$ (bistable medium). In addition to stationary, oscillatory, and Turing patterns, quasiperiodic inhomogeneous oscillations and spatiotemporal chaos were also found. States belonging to the homogeneous limit cycle, which exist for the case $\gamma = 0.9$, were considered as initial conditions. This choice corresponds to the idea of having an initially homogeneous oscillatory medium which is suddenly modified in a localized region (for example, by lighting photosensitive chemical reactions in gels [10]). In the central bistable region, the fields rapidly converge to the values ($u_\pm \simeq \pm 0.8, v_\pm \simeq \pm 0.14$) of a homogeneous bistable system with $\gamma = \gamma(x=0) = 5.9$ ($\pm$ depending on the initial condition [11]), while the rest of the system evolves to different asymptotic behaviors. Note that Eqs. (1) have an odd symmetry, implying that if $[u(x,t), v(x,t)]$ is a solution, then $[-u(x,t), -v(x,t)]$ is a solution as well. In Fig. 1 we show the phase diagram indicating the asymptotic behavior of the system [stationary patterns (SP's), Turing (TP), inhomogeneous periodic oscillations (PO's), quasiperiodic (QP), and chaotic (CH)] as functions of the parameters $D_v$ and $L$. Here it is worth remarking that the error in the determination of the position of the transition lines is of the order of the width of the plotted lines in the picture.

What we here call stationary patterns appear for small $L$ (smaller than two Turing wavelengths of the medium [4]) as a consequence of the relative large size of the bistable spot that prevents the system from asymptotically performing temporal oscillations. The Turing patterns, that are also stationary, have a completely different origin (they are a con-

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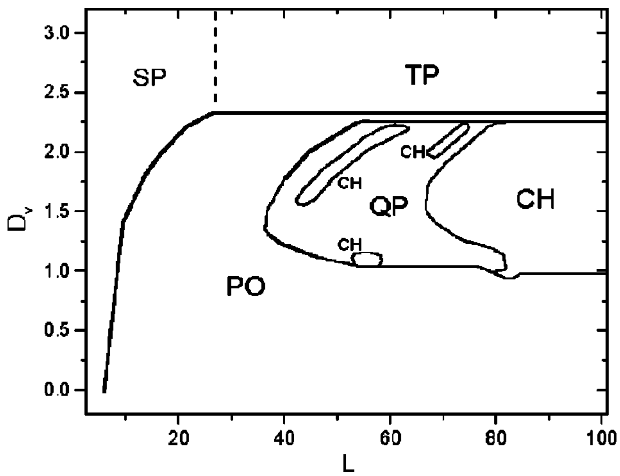


FIG. 1. Phase diagram $[(L,D_v)$ plane$]$ corresponding to a one-dimensional inhomogeneous system consisting of an oscillatory medium with a central bistable spot. The regions corresponding to stationary patterns, periodic oscillations, Turing patterns, and quasiperiodic and chaotic behaviors are indicated by SP, PO, TP, QP, and CH, respectively.

sequence of the Turing diffusional instability of the medium), and arise for larger systems [4].

In this work we continue the line of research started in Ref. [4], analyzing more realistic bidimensional systems combining oscillatory and bistable domains. The aim of this paper is to present, in a descriptive way, some examples of inhomogeneous situations in bidimensional systems leading to different kinds of complex dynamics. We do not intend here to describe all the possible complex dynamics that could arise in the kinds of systems studied. Instead, we want to show that a wide spectrum of possibilities appears when some kind of inhomogeneous situations is considered. Our goal will be to attract the interest of experimentalists to study inhomogeneous pattern forming systems with characteristics similar to those here analyzed.

We will extend the previous theoretical analysis to some situations that are generalizations of the unidimensional case studied in Ref. [4]: a square oscillatory medium with a central circular bistable spot, a rectangular oscillatory medium with a bistable stripe and inhomogeneous diffusion, and a trapezoidal oscillatory medium with a bistable stripe. The first case is the simplest of all of these. However, such a system exhibits some new interesting dynamical characteristics. The other two cases are nontrivial generalizations of the system discussed in Ref. [4]. The case of the rectangular system corresponds to the coupling between unidimensional systems with different inhibitor diffusions (that is, sweeping in a vertical direction in the phase diagram of Fig. 1). The case of the trapezoidal system corresponds to the coupling between unidimensional systems with different lengths (that is, sweeping in a horizontal direction in the phase diagram of Fig. 1). In this way we expect to see the result of the mixing, in a bidimensional system, of the different behaviors (phases) occurring in the swept region of the phase diagram of the unidimensional system. Hence we will be studying examples of pattern forming systems that couple, say, chaotic regions, with others that are, for instance, periodic or Turing-like.

In this paper we will work with finite systems. We want to remark here that the size of the system is always a relevant parameter in our studies, as the asymptotic regime depends on this. Also, boundary conditions are important in determining the dynamics: all the results in this paper are for nonflux (Neumann) boundary conditions, which is the usual choice when dealing with chemical systems. However, some numerical studies, that are in progress, have shown us that similar results may be obtained using more general partially reflective conditions $[\partial u(L)/\partial x = ku(L)]$ for small enough values of $k$, the albedo (or reflectivity) parameter.

Similarly to what was done in Ref. [4], in the three problems studied in this work we will consider homogeneous initial conditions belonging to the limit cycle existing for an homogeneous system with $\gamma = 0.9$. As explained above for the case of a unidimensional system, the inhomogeneities of the media are assumed to be caused by an external mechanism, and to appear in a sudden way when the system (which is originally purely oscillatory) is performing an homogeneous periodic motion.

The experimental observation of the predicted behaviors might be realized by adequately designing chemical or electrical systems [12] sharing the properties of the models here discussed. One simple option seems to work with chemical systems. The idea is to prepare the system (for instance Belousov-Zhabotinskii or CIMA-like reactions) with photosensitive catalyzers or reactants, allowing us to create inhomogeneities by illuminating some adequately selected areas [13]. Another option is to work with inhomogeneous gels providing, for example, a way to obtain a inhomogeneous diffusivity [14]. Clearly, it is also possible to explore mixed situations.

All numerical calculations have been made as follows. First, different systems of partial differential equations were approximated by systems of coupled ordinary differential equations, obtained by finite difference schemes. Second, the resulting equations were solved by a Runge-Kutta 2 method. Different space and time discretizations schemes were employed in order to check the results.

The details of the functional forms chosen for introducing the inhomogeneities in $\gamma$—in the one-dimensional problem discussed in Ref. [4] and also in the bidimensional problems analyzed in the following sections—are irrelevant, as extensively checked in simulations. The relevant fact is the choice of the coupling between bistable and oscillatory regions (with sharp interfaces). Furthermore, in the unidimensional problem, even the size of the bistable spot is unimportant as, for example, an increase of this parameter causes only a slight shift of the transitions lines but preserves all the structure of the phase diagram unchanged. This is because, in the unidimensional problem, the bistable region acts almost as an effective boundary condition. The robustness of the results with respect to changes in the functional form of $\gamma(x)$ also becomes apparent as, when trying different spatial discretizations (that cause the function to be evaluated in different positions), the results remains essentially unchanged. All the calculations for the bidimensional systems analyzed in

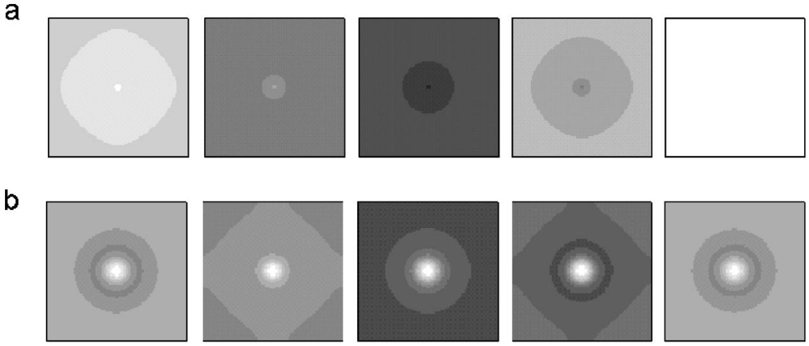


FIG. 2. Contour plots of the $u$ field for five different times during a period of the motion ocurring for (a) $D_v=0$, $L=25$, and $r_0=0.5$ (symmetric periodic oscillation); and (b) $D_v=1$, $L=25$, and $r_0=2.5$ (asymmetric periodic oscillation). In both cases, from left to right, the plots correspond to times $t_0$, $t_0+\tau/4$, $t_0+\tau/2$, $t_0+3\tau/4$, and $t_0+\tau$. With $t_0$ in the asymptotic regime and $\tau$ the measured period of the motion.

this paper have been done using much finer spatial discretizations than the one needed to clearly identify the kind of regime appearing in the system for the values of parameters considered.

The organization of the paper is as follows: in Sec. II we study the pattern formation in a square oscillatory medium with a circular bistable spot. We also explain the method used to identify the kind of behavior occurring in the system (chaotic or quasiperiodic). In Sec. III we analyze the case of a rectangular reactor with inhomogeneous diffusion of the inhibitor while in Sec. IV we present results for a trapezoidal reactor. Section V is devoted to a discussion of results and to some conclusions.

## II. SQUARE OSCILLATORY MEDIUM WITH A CIRCULAR BISTABLE SPOT

Here we consider a nonlinear medium described by Eqs. (1) in a geometry corresponding to a square domain ($-L \leqslant x \leqslant L, -L \leqslant y \leqslant L$) with nonflux boundary conditions. We introduce a spatial dependence for parameter $\gamma$, setting $\gamma(x,y)=\gamma(r)=0.9+2.5[1+\tanh(-6(r-r_0))]$ [15], where $r=\sqrt{x^2+y^2}$. As the system is bistable for $\gamma>1$ and oscillatory for $\gamma<1$, with this choice of $\gamma(r)$ the medium is oscillatory (with $\gamma \simeq 0.9$) except in a central spot of radius $r \simeq r_0$ where it is bistable (with $\gamma \simeq 5.9$). As in the unidimensional problem described in Ref. [4] (a one-dimensional system), we consider a homogeneous initial condition belonging to the homogeneous limit cycle of a purely oscillatory medium (with $\gamma=0.9$). We have solved Eqs. (1) numerically for different values of $L, D_v$, and $r_0$.

The results for the bistable spot in two dimensions show a similar variety of behaviors as in one dimension (stationary, Turing, periodic, quasiperiodic, and chaotic patterns). However, here we have one important difference added to the complexity of the dynamics: the existence of asymptotic motions where $u(0,0)$ and $v(0,0)$ oscillate symmetrically about zero, instead of setting near one of the two states of the bistable media $(u_\pm, v_\pm)$, as occurs in the one-dimensional (1D) system. For some region of parameters, such oscillations can be interpreted as periodic (quasiperiodic or chaotic) transitions between the two states of the bistable spot, as $u(0,0)$ [$v(0,0)$] oscillates symmetrically around 0 with an amplitude similar to (but slightly lower than) $u_+(v_+)$. In other cases, the symmetrical oscillations of the bistable spot have an amplitude much smaller than $u_+$ and every remaining trait of bistability is lost. This phenomenon occurs for small enough values of $r_0$ and $D_v$, and can be interpreted as follows: in the case of strong coupling or large bistable region (high $D_v$ or $r_0$) the influence of the surrounding oscillatory medium is not strong enough to produce transitions between the two states of the bistable region and the fields adopt values near $(u_\pm, v_\pm)$. In the opposite case (small enough $r_0$ or $D_v$) the influence of the surroundings is stronger and within the bistable region the fields oscillate around zero.

Figure 2 shows contour plots of the $u$ field corresponding to different times during a period of the asymptotic periodic regime of the system for two different sets of parameters. Figure 2(a) shows the symmetric oscillation of the bistable spot (the center oscillates from white to black during a period), while in Fig. 2(b) the central spot remains in one of the two states of the bistable medium. We call the asymptotic regime symmetric or asymmetric depending on whether the bistable spot behaves as in Fig. 2(a) or 2(b), respectively.

It is worth mentioning that for the 1D system, even for very small values of $r_0$ (i.e., $r_0=0.1$), the regime is always asymmetric (the bistable spot is always fixed in one of its two states). Furthermore, for $r_0=0.05$ we observed a transition in the asymptotic regime from periodic to quasiperiodic (for example for $D_v=2$ and $L=40$) that remains asymmetric. This important difference between the behavior of the 1D system and the bidimensional system shows the stronger influence of the surrounding oscillatory medium on the bistable spot in the bidimensional case.

As in the 1D system, two different stationary regimes appear: for small values of $L/r_0$, stationary patterns appear as a consequence of the relatively large size of the bistable domain [4], while for higher values of $L$ and $D_v$ Turing patterns arise. Figure 3 shows patterns corresponding to two different sets of parameters in the Turing region. The Turing instability appears in the oscillatory medium for $D_v>1.7$, and coexists with the Hopf instability [4,16]. Roughly for $D_v>2.3$ the Turing instability completely dominates the dynamics, that is, the system evolves to stationary spatially periodic patterns. These patterns are generated by freezing fronts [4,12] that propagate from the bistable region to the rest of the system, changing the dynamics from Hopf-like to Turing-like. The characteristic Turing wavelength is $\lambda_T \simeq 13$.

In the 1D system [4], the nonstationary and nonperiodic asymptotic regimes are classified as quasiperiodic or chaotic

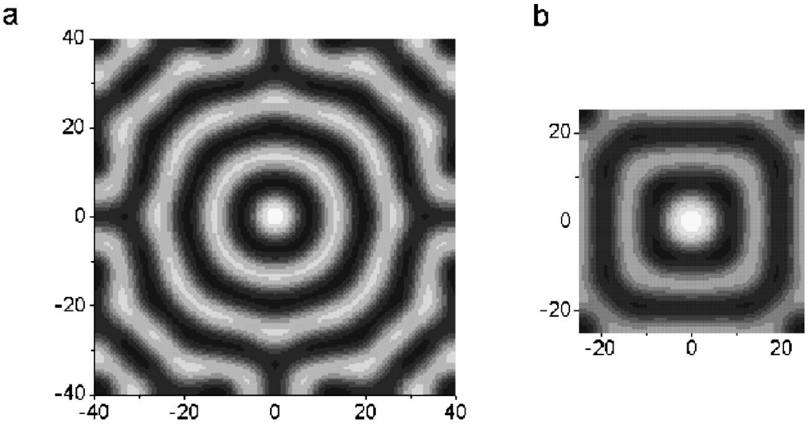


FIG. 3. Turing patterns appearing for (a) $D_v = 2.5$, $L = 40$, and $r_0 = 0.5$; and (b) $D_v = 2.5$, $L = 25$, and $r_0 = 2.5$.

by analyzing the sensibility to initial conditions. In the bidimensional problems such a study involves a much larger computational cost. Here, as a cheaper alternative, we present a method that allows us to determine whether the behavior is chaotic or quasiperiodic by simple glancing at a discrete temporal series. At a given spatial position $(x,y)$ we consider the succession $\{t_n(x,y)\}$ of times at which $u(x,y,t)$ reaches a local maximum as function of $t$, i.e., the times when

$$\frac{\partial}{\partial t} u(x,y,t)|_{t_n(x,y)} = 0 \qquad (2)$$

and

$$\frac{\partial^2}{\partial t^2} u(x,y,t)|_{t_n(x,y)} < 0 \qquad (3)$$

occur simultaneously. Then we define a succession

$$\{p_n(x,y) = t_n(x,y) - t_{n-1}(x,y)\}. \qquad (4)$$

We first studied the behavior of $\{p_n(x=L)\}$ for the 1D system, finding that this analysis provides a useful and simple way to determine the character of the dynamics (periodic, quasiperiodic, or chaotic). In the case of periodic motion the value of $p_n(L)$ converges to a constant $p_\infty$ which coincides with the period of the global motion. In the case of quasiperiodic motion, the $p_n(L)$'s asymptotically show a periodic or quasiperiodic behavior. In the chaotic region, which was defined originally as the region with a high sensitivity to initial conditions, the $p_n(L)$'s exhibit a highly disordered behavior. In Fig. 4 we show the typical plots of the $p_n(L)$'s in all three regions.

For the bidimensional problem discussed in this section, we have studied the plots of $\{p_n(L,L)\}$, finding the same three types of behavior as in the 1D system. In the cases where $p_n(L,L)$ converges to a constant, we have verified that, as in the one-dimensional case, the global motion is periodic with a period equal to that constant. We have classified the nonperiodic motions as quasiperiodic or chaotic depending on whether the behavior of the $p_n(L,L)$'s is like that in Fig. 4(b) or Fig. 4(c), respectively. A remarkable fact here is that we have not found marginal or confused cases where the distinction between the types of behavior is not completely clear.

In Table I we present the results of some of the calculations for different values of the parameters $L, r_0$, and $D_v$, indicating the kind of asymptotic motion by the same nomenclature as in Fig. 1 and adding $S$ or $A$ according to whether the oscillations are symmetric or asymmetric, respectively, in the sense discussed above. In the table, the following tendencies can be appreciated, some of them being analogous to those appearing in the 1D system, which are clearly reflected in the phase diagram of Fig. 1.

(i) For small values of $L$, only stationary or periodic behaviors are observed. The tendency to stationarity introduced by the presence of the inhomogeneity causes the amplitude and frequency of the oscillations in the oscillatory media to decrease. For very small $L$ the oscillations are completely inhibited in the whole system; for larger $L$, an inhomogeneous periodic regime (with a frequency slightly lower than the natural of the oscillatory media) is asymptotically reached.

(ii) For small $D_v$ ($\leqslant 1$) the behavior is periodic (or stationary for very small L). The higher $D_v$ is the richer the spatial structure of the patterns in the PO region, due to the more efficient transmission of the effect of the inhomogeneity.

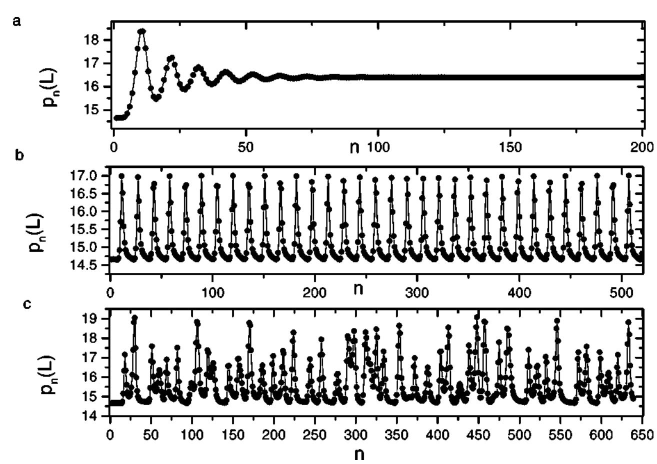


FIG. 4. Plots of $p_n(L)$ for a 1D system corresponding to (a) $D_v = 1.7$, $L = 35$ (periodic oscillations), (b) $D_v = 1.7$, $L = 50$ (quasiperiodic motion), and (c) $D_v = 1.7$, $L = 80$ (chaos).

TABLE I. Results of the calculations for different values of the parameters $L$,$r_0$, and $D_v$. Here we indicate the kind of asymptotic motion using the same nomenclature as in Fig. 1. We add *S* or *A* according to whether the oscillations are symmetric or asymmetric, respectively.

| $D_v$ ↓ | $L=8$ $r_0=0.5$ | $L=8$ $r_0=0.25$ | $L=25$ $r_0=0.5$ | $L=25$ $r_0=2.5$ | $L=40$ $r_0=0.5$ | $L=40$ $r_0=2.5$ | $L=60$ $r_0=0.5$ | $L=60$ $r_0=2.5$ |
|---|---|---|---|---|---|---|---|---|
| 0 | PO-*S* | PO-*A* | PO-*S* | PO-*A* | PO-*S* | PO-*A* | | |
| 1 | PO-*S* | SP | PO-*S* | PO-*A* | PO-*S* | PO-*A* | | |
| 1.5 | PO-*S* | SP | QP-*S* | PO-*A* | QP-*S* | QP-*A* | CH-*S* | CH-*S* |
| 2 | PO-*S* | SP | PO-*A* | PO-*A* | CH-*S* | QP-*A* | CH-*S* | CH-*S* |
| 2.5 | SP | SP | PO-*A* | TP | TP | TP | | |

(iii) There exists a region of intermediate values of $D_v$ (approximately $1<D_v<2.5$ depending on $r_0$) where quasiperiodic or chaotic motion may appear. Note that the value $D_v=1$ is a precise limit in the one-dimensional case (see Fig. 1) for the appearance of nonperiodic behavior. This limit corresponds to the diffusion of the inhibitor being equal to that of the activator, the last one being fixed equal to 1 in Eq. (1).

(iv) For small $r_0$ or small $D_v$, the bistable spot oscillates symmetrically around 0, as explained above.

(v) For high $D_v$ (approximately $\geqslant 2.5$), Turing patterns are generated by means of freezing fronts.

(vi) A reduction in the spot's size can, at least in some cases, increase the complexity of the dynamics. (For example, causing changes from stationary to periodic, periodic to quasiperiodic or from quasiperiodic to chaotic regimes.)

## III. A RECTANGULAR REACTOR WITH INHOMOGENEOUS DIFFUSION OF THE INHIBITOR

Here we consider the case of a rectangular oscillatory medium ($-L_x \leqslant x \leqslant L_x, -L_y \leqslant y \leqslant L_y$) with a bistable stripe around $y=0$. In this case we solve Eqs. (1) with $\gamma(x,y)=\gamma(y)=0.9+2.5[1+\tanh(-6(y-y_0))]$, with no flux boundary condition and homogeneous initial conditions as before. Here $y_0$ is (approximately) the half width of the bistable stripe. For homogeneous initial conditions the system would be equivalent to the one-dimensional case discussed in Ref. [4]. However, here we consider an additional inhomogeneity in the medium: a dependence of the inhibitor diffusion $D_v$ on the $x$ coordinate. We set

$$D_v = D_{v0} + (x+L_x)\frac{(D_{v1}-D_{v0})}{2L_x}, \tag{5}$$

with $D_{v0}<D_{v1}$.

Now, the bidimensional system (2D system) can be thought as a (continuous) array of coupled 1D systems analyzed in Ref. [4], one for each value of $x$ with a value of $L=L_y$ and $D_v=D_v(x)$. This 2D system can be associated with a vertical segment in the phase diagram of Fig. 1, going from $(L_y, D_{v0})$ to $(L_y, D_{v1})$. This segment may cross one or more transition lines separating regions of different asymptotic behavior of the 1D system. In such cases, different parts of the 2D system will tend to perform different asymptotic motions. The global long time behavior of the 2D system will emerge as the result of the coupling and competition between the different regions, and will depend not only on the regions involved but also on their sizes.

Here we present and discuss the results for five different 2D systems associated with the five vertical segments indicated with Figs. 5(a)–5(c). In all the cases we have used the value $L_x=15$.

First we study the example corresponding to a case where the 2D system is in the periodic region of the phase diagram of Fig. 5(a) for all values of $x$. The parameters are $L_y=30$, $D_v(-L_x)=D_{v0}=0$, and $D_v(L_x)=D_{v1}=2$. The global behavior of the system is periodic. In Fig. 6(a) we show the contour plots of the $u$ field corresponding to different times during a period of motion. It can be observed that for $x$ near $-L_x$ (small $D_v$) the oscillation is almost homogeneous in the $y$ coordinate, while near $L_x$ (larger $D_v$) the spatial structure is more complex. This can be better appreciated in Fig. 6(b), where we show space time plots $(t,y)$ of the $u$ fields for $x=-L_x$ and $x=L_x$. In the first case, the oscillations are homogeneous, while in the latter there are periodic waves that travel from $\pm L_y$ toward the bistable region. These properties are inherited from the characteristics of the 1D system for the corresponding values of $D_v$, and are similar to those reported in Sec. II the case of the square oscillatory medium with the circular bistable spot.

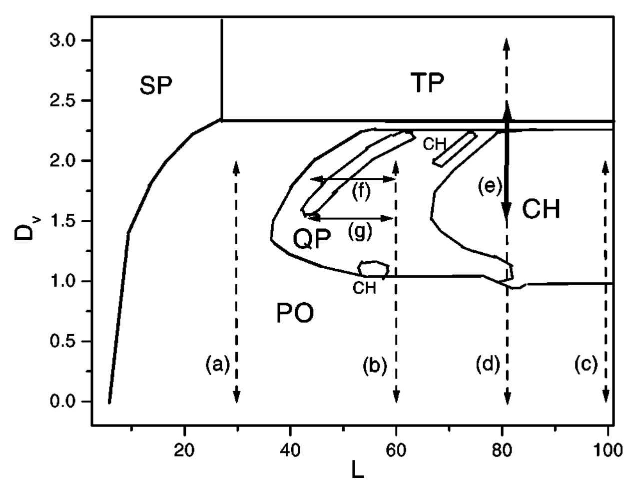


FIG. 5. Phase diagram [$(L,D_v)$ plane] corresponding to the 1D system studied in Ref. [4], indicating the segments associated with different cases of the bidimensional problem analyzed in Secs. III and IV.

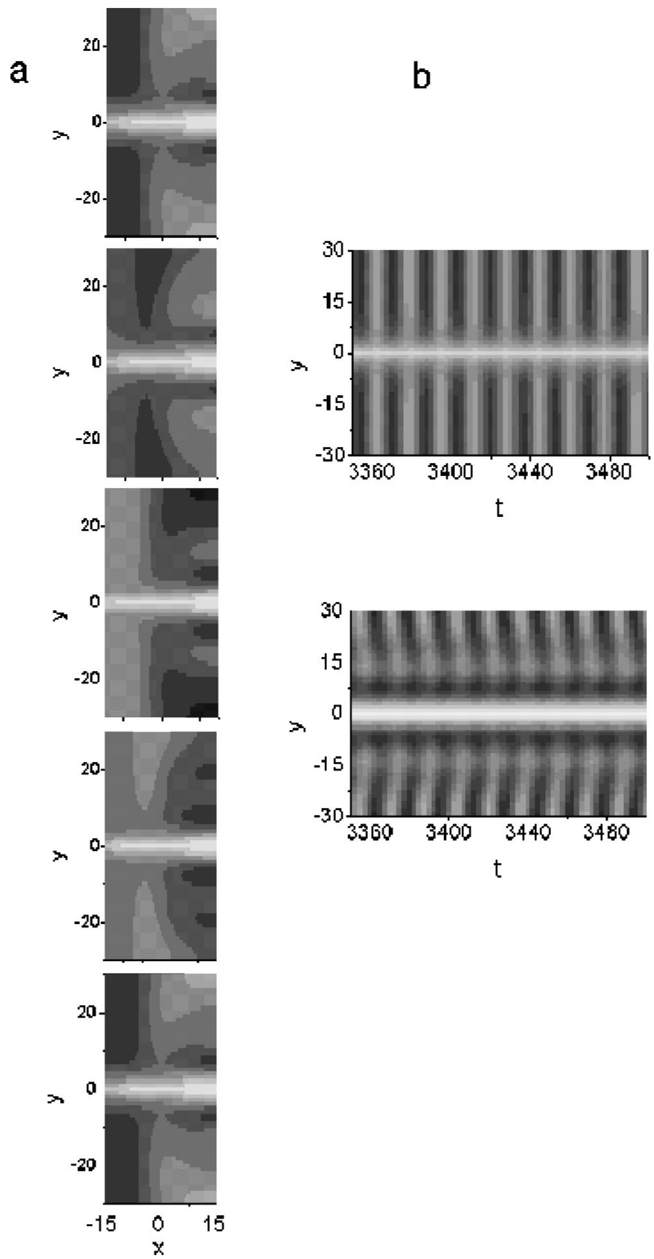


FIG. 6. (a) Contour plots of the $u$ field for five different times during a period of the (periodic) motion ocurring for $D_{v0}=0$, $D_{v1}=2$, and $L_y=30$. From top to bottom, the plots correspond to times $t_0$, $t_0+\tau/4$, $t_0+\tau/2$, $t_0+3\tau/4$, and $t_0+\tau$. With $t_0$ in the asymptotic regime and $\tau=16.42$, the measured period of the motion. (b) Space-time plots of the $u$ field at $x=-15$ (top) and $x=15$ (bottom) for the same asymptotically periodic motion.

The second system analyzed [corresponding to the vertical segment (b) in Fig. 5] is in the quasiperiodic region for $x>0(D_v>1)$ and in the periodic region for $x<0$. The parameters are $L_y=60$, $D_v(-L_x)=D_{v0}=0$, and $D_v(L_x)=D_{v1}=2$. In this case the global behavior is quasiperiodic. In Fig. 7(a) we show space time plots of the $u$ field for $x=-L_x$, $x=0$, and $x=L_x$. In the first case we note that homogeneous oscillations (in the $y$ coordinate) alternate with traveling waves going from $y=\pm L_x$ to the bistable zone. Around $x=0$ the behavior is similar, but there are periods of time in which waves emerging from near $y=\pm L_y/2$ appear. For $x=L_x$, waves emerging from the bistable zone that are annihilated near $y=\pm L_y/2$, with others coming from $y=\pm L_y$ can be observed. Furthermore, the typical signature of quasiperiodicity and chaos in the space time plot, which is the onset of defects (dislocations), is also present. In this case, the increase of complexity originated by the increase of $D_v$—also observed and explained in the previously studied systems—is apparent.

Third, we analyze the case corresponding to the parameters $L_y=100$, $D_v(-L_x)=D_{v0}=0$, and $D_v(L_x)=D_{v1}=2$ [segment (c) in Fig. 5]. For $x<0$ the system is in the periodic region, while for $x>0$ it is in the chaotic one. In this case the global behavior is chaotic. The different kinds of waves appearing in the space time plots (not shown) are similar to those of Fig. 5(b). However, in this case the dislocations appear not only for $x>0$ but also for $x<0$. This means that the signature of chaos have invaded the whole system.

For the case of Fig. 5(d) [$L_y=80$, $D_v(-L_x)=D_{v0}=0$, and $D_v(L_x)=D_{v1}=3$], the system has approximately one-third of its area in the periodic region, one-third the chaotic region, and one-third in the Turing region. The global behavior is periodic. This is an unexpected result, since the system has a significant part of its area within the chaotic region. Unlike Fig. 5(c), where the chaotic behavior advances over the periodic zone, here the periodic and stationary (Turing) tendencies of the corresponding zones cooperate to inhibit the chaotic features in the system. This case is a clear example of coexistence of Turing- and Hopf-like domains [16,12]. In Fig. 7(b), we show the contour plots of the $u$ fields corresponding to two different times during a period of motion. It is apparent here that the Turing pattern appearing on the right side of the system acts effectively as a boundary condition, limiting the chaotic tendency of the central part of the system.

For the case of segment (e) in Fig. 5 [$L_y=80$, $D_v(-L_x)=D_{v0}=1.5$, and $D_v(L_x)=D_{v1}=2.4$] the system has approximately one-third of its area in the Turing region, while the rest is chaotic. Now the global behavior is also chaotic. In this case, the weight of the Turing zone is not enough to prevent the chaotic behavior. A very interesting phenomenon occurs here: in the Turing zone, the Turing patterns appear and disappear repeatedly, alternating with quasi-homogeneous oscillations. The Turing pattern is generated by freezing fronts like those described in Ref. [4], that come from the bistable region, live for a certain period of time (typically of the order of $20\tau_0$), and then are destroyed by ''melting'' fronts going from $y=\pm L_y$ to the bistable stripe. In Fig. 8 we show a space time plot exhibiting this phenomenon.

The five cases discussed in this section clearly illustrates the wide spectrum of possibilities that arises when considering inhomogeneous situations where regions of different ''natural'' behaviors are coupled. Such systems, beyond the problem of combination of bistable and oscillatory media, are in fact examples of pattern forming systems that couple regions showing chaotic behavior with others that have a periodic or Turing-like behavior. In Sec. IV we will continue

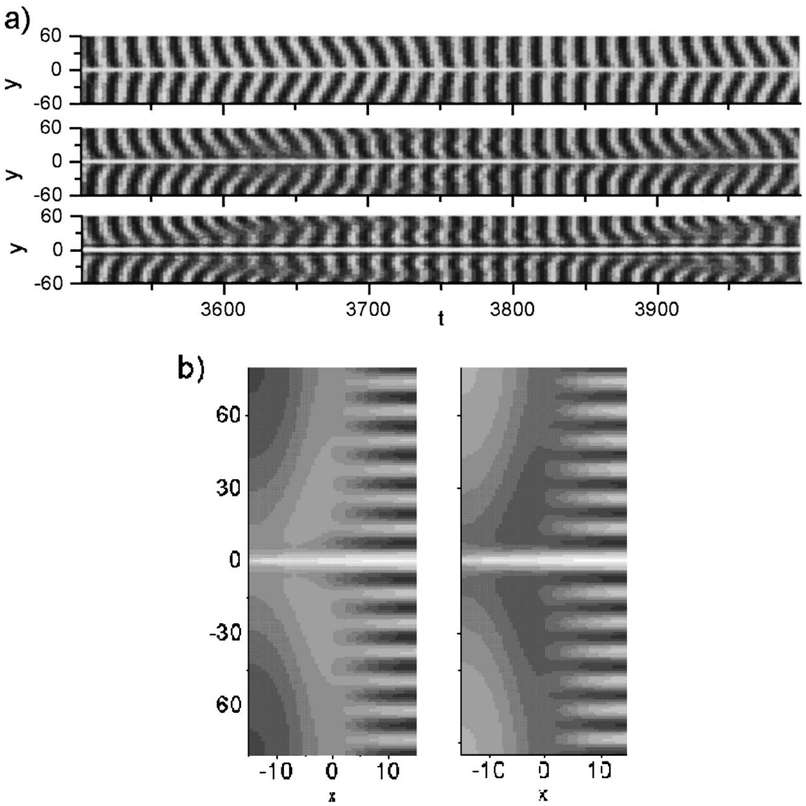


FIG. 7. (a) Space-time plots for the $u$ field corresponding to the system with $L_y=60$, $D_v(-L_x)=0$, and $D_v(L_x)=2$. The plots are for $x=-15$, $x=0$, and $x=15$ from top to bottom. (b) Contour plots of the $u$ field for two different times (differing in half period) in the periodic motion ocurring for $D_{v0}=0$, $D_{v1}=2$, and $L_y=30$. Calculations correspond to Fig. 5(d).

with this general view, analyzing a different inhomogeneous situation in a trapezoidal reactor.

## IV. A TRAPEZOIDAL REACTOR

Here we present some results for the case of a trapezoidal reactor containing an oscillatory medium with a bistable stripe. We consider a system with $0\leq x\leq L_x$ and $L_y=L_y(x)=L_0+ax$, fixing Neumann boundary conditions and uniform initial conditions as usual. The inhomogeneity is the same as in the problem of Sec. III, given by $\gamma(x,y)=\gamma(y)=0.9+2.5[1+\tanh(-6(y-y_0))]$, implying a bistable stripe of a half width approximately equal to $y_0$ around $y=0$. The rest of the medium is oscillatory with $\gamma\simeq 0.9$.

In this case, the system can be thought as a (continuous) array of coupled 1D systems (one for each value of $x$), each one with a different length (since $L_y$ depends on $x$), and a uniform value of $D_v$. Similarly to the case analyzed in Sec. III, with this 2D system we can associate a now horizontal segment in the phase diagram of Fig. 1, going from $[L_y(0),D_v)$ to $(L_y(L_x),D_v]$, that may cross one or more transition lines. The system shows different asymptotic global behavior depending on $L_y(0)$, $a$, and $D_v$, that can be (as in the other analyzed systems) stationary, periodic, quasiperiodic, chaotic or Turing-like. In Fig. 9 we show the contour plots of the $u$ field at two different times in the asymptotic regime for a case in which the behavior is chaotic.

Here we report the analysis of two particular examples corresponding to two different horizontal segments in the quasiperiodic region of the phase diagram of the 1D system [which are shown in Figs. 5(f) and 5(g)]. In both cases we have fixed $L_x=30$. The segment in Fig. 5(f) $[L_y(0)=45$, $D_v=1.9$, and $a=1/3]$ crosses a small chaotic window,

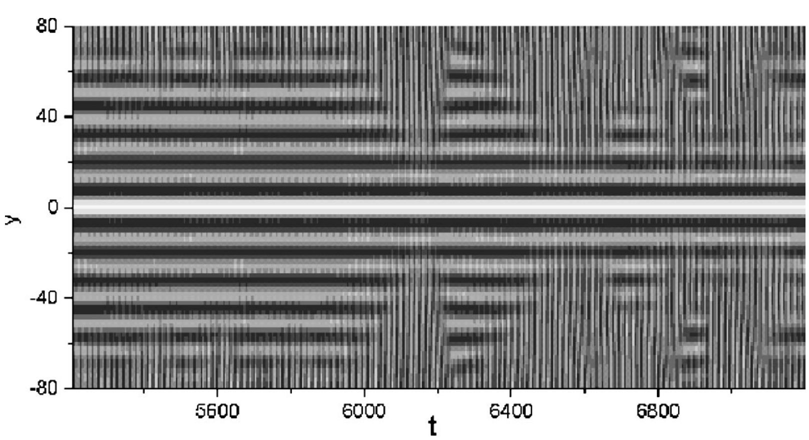


FIG. 8. Space-time plot of the $u$ field for $x=L_x$, corresponding to the system with $L_y=80$, $D_v(-L_x)=1.5$, and $D_v(L_x)=2$. We observe the intermittent appearing and disappearing of the Turing pattern.

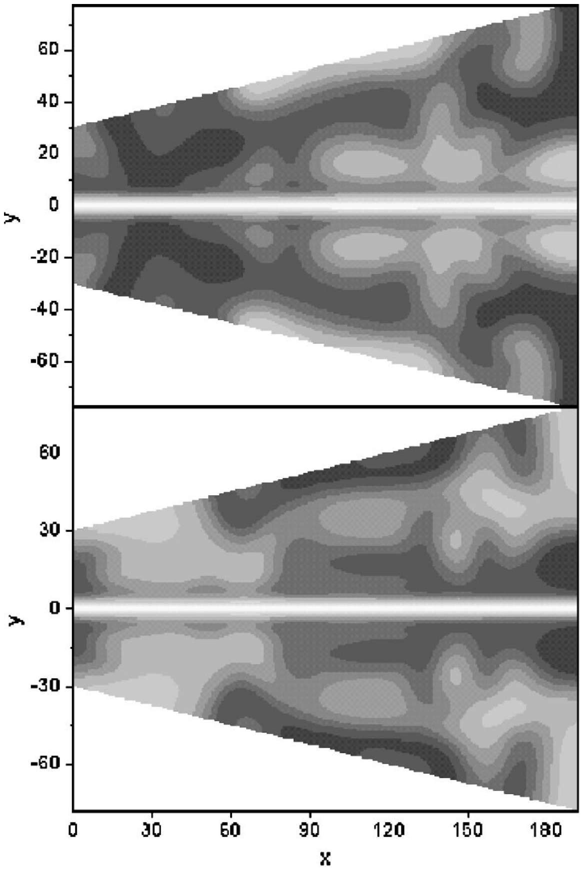


FIG. 9. Contour plots of the $u$ field for a trapezoidal system at two different times (differing approximately by half the characteristic period of the oscillatory medium with $\gamma=0.9$). The parameters are $L_x=192$, $L_y(x=0)=30$, $a=0.25$, and $D_v=1.3$.

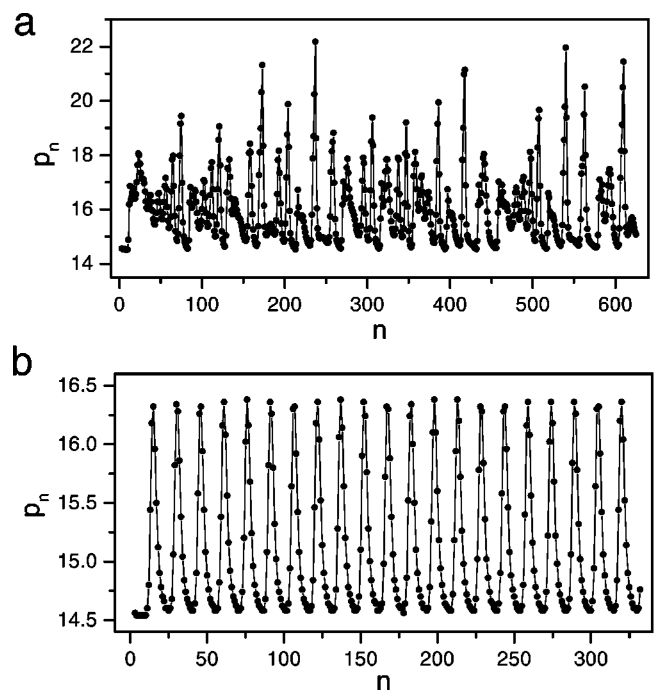


FIG. 10. Plots of $p_n[L_x,L_y(Lx)]$ for trapezoidal systems corresponding to (a) Fig. 5(f) and (b) Fig. 5(g).

while the segment of Fig. 5(g) $[L_y(0)=45$, $D_v=1.5$, and $a=1/3]$ lies completely within the quasiperiodic region of the phase diagram. In Fig. 10 we show the plots of $p_n[L_x,L_y(Lx)]$ for both cases, from which it can be seen that the behavior is chaotic in the case of the Fig. 5(f) and quasiperiodic in the case of Fig. 5(g). It is worth remarking here that in the first case the presence of the small chaotic window is enough to induce the chaotic behavior in the whole system. Note that the signal in Fig. 10(a) is measured in a position [the right end of Fig. 5(f)] where, according to the original diagram, it should behave as a quasiperiodic (nonchaotic) one.

## V. FINAL REMARKS

We have presented three different examples of bidimensional inhomogeneous pattern-forming systems, each consisting of an oscillatory medium with a bistable domain, in different geometries with Neumann boundary conditions. The work extends the studies started in Ref. [4], where a simple unidimensional inhomogeneous system was analyzed. Such systems correspond to examples of pattern-forming systems coupling regions where the dynamics is chaotic, with others where the dynamics is periodic or Turing-like. We expect that the results obtained here should be ubiquitous for inhomogeneous media having the geometrical distribution of bistable and oscillatory regions discussed here, and would not depend on the specific model equations considered.

Due to the expected connection with experimental setups using photosensitive catalyzers, we have focused on a homogeneous initial condition, as this is one of the most relevant cases. The choice of other kinds of initial conditions may cause the asymptotic regime to be different from those found for homogeneous initial conditions, and may also lead to regimes not predicted here. A more general analysis of problems with arbitrary initial conditions is clearly beyond the scope of this paper.

In the first example we studied, which corresponds to the case of a square oscillatory medium with a central circular bistable domain, we observed an important difference from the results of the unidimensional system found in Ref. [4]. This difference appears in the possibility of observing jumps between the two states of the bistable domain, which are induced by the surrounding oscillatory medium or, equivalently, a symmetric oscillation of the fields in the bistable domain around the zero value. Apart from this phenomenon, we again found the same diversity of behavior (including Turing patterns, spatiotemporal chaos, etc.) as in the unidimensional system.

As explained in Sec. I, referring to the unidimensional case, Eqs. (1) have odd symmetry, implying that if $[u(x,t),v(x,t)]$ is a solution, then $[-u(x,t),-v(x,t)]$ is a solution as well. This symmetry is clearly also present in the bidimensional case. For those cases where the bistable central region converges to one of its stable states (all the studied cases with the exception of the symmetric motions in Sec. II), a general bistability behavior exists.

The second example we studied corresponds to a rectangular oscillatory medium with a bistable stripe with inhomogeneous diffusion of the inhibitor (varying linearly along the direction of the stripe). In this case (at least for the widths of the stripe considered), the bistable domain remains fixed in one of the two possible states. Depending on the values of the minimum and maximum diffusion of the inhibitor, and on the length of the system in the direction normal to the bistable stripe, we found different kinds of global behavior (again including and quasiperiodic oscillations, chaos, and stationary—Turing—patterns). We have analyzed the different spatiotemporal waves appearing, viewing the system as a set of coupled unidimensional systems like the one studied in Ref. [4]. Among other phenomena, we found the coexistence of Turing- (in the region of high $D_v$) and Hopf-like (in the region with small $D_v$) [16,12] behaviors in a globally periodic regime; and the consequent production and annihilation of a Turing pattern (which alternates with temporal inhomogeneous oscillations) occurring in a globally chaotic system.

The third example we studied corresponds to a trapezoidal oscillatory medium with a bistable stripe that can be seen as a coupling between unidimensional systems with the same inhibitor diffusion but different lengths. The existence of different complex behaviors is also apparent. In particular, we have seen that sweeping through a small chaotic window may be enough to induce chaotic behavior in the whole system.

In all the analyzed examples (and also in the unidimensional problem referred to in Sec. I), the Turing patterns are generated by means of freezing fronts that travels from the bistable domain to the rest of the system, changing the dynamics of the oscillatory media from Hopf-like to Turing-like. The spatially periodic patterns generated in such a way have (essentially) a single wave vector which is normal to the boundary of the bistable domain (parallel to the direction of propagation of the freezing front). However, in the problem studied in Sec. II (see Fig. 3), the Turing patterns does not have the symmetry of the circular bistable spot, as a consequence of the square shape of the system and the Neumann boundary conditions.

To distinguish between chaotic and quasiperiodic behaviors, we have introduced a simple method in which we analyze a temporal signal taken from a unique spatial position. In this way we avoid the calculation of the Lyapunov exponents, which implies a large computational cost for the bidimensional systems.

It is also worth mentioning here that in this work we have focused on solutions with some spatial symmetry. For example, in the cases discussed in Secs. III and IV, the solutions have even parity in the $y$ coordinate. For some particular sets of parameters within the chaotic regime, we have observed a numerical breaking [17] of this symmetry. In the studied cases, these effects (that are common when using finite difference schemes for solving reaction diffusion equations) were suppressed by improving the spatial discretization, though at the price of increasing the computational time. A less time consuming option for further analysis of the chaotic dynamics in the systems analyzed here could be provided by pseudospectral methods [17]. A richer characterization of the chaotic behavior in this kind of system, exploiting the biorthogonal decomposition method [18] as well as some topological analysis of chaotic time series [19], will be the subject of further work.

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